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Mathematics

Exhaustion functions and Stein neighborhoods for smooth pseudoconvex domains

(Serre conjecture/holomorphic convexity/Runge pairs)

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ABSTRACT A strictly plurisubharmonic exhaustion function with negative values is constructed for arbitrary relatively compact pseudoconvex domains with smooth boundary in a Stein manifold. It is applied to verify the Serre conjecture in a special case. A sufficient condition is given that guarantees the existence of a neighborhood-basis of Stein domains for certain bounded pseudoconvex domains on a Stein manifold.

It is well known that any strictly pseudoconvex bounded domain Ω in $\mathbf{C}^n$ has the following two properties:

(*1*) There exists a strictly plurisubharmonic C^∞-function ρ on Ω with negative values and

$$\lim_{z \to b\Omega} \rho(z) = 0.$$

(*2*) There exists a neighborhood basis for $\overline{\Omega}$ consisting of strictly pseudoconvex domains.

It is not known which bounded (weakly) pseudoconvex domains Ω in $\mathbf{C}^n$ have these properties. The purpose of this note is to announce some results containing sufficient conditions on Ω for the properties (*1*) and (*2*) to hold. The detailed proofs of these results will appear in a later paper of the authors.

Notations and examples

Definition: Let Ω be a domain in $\mathbf{C}^n$ (or on an n-dimensional complex manifold). A (strictly) plurisubharmonic function ρ is called a (strictly) plurisubharmonic exhaustion function of Ω, if it defines a proper mapping of Ω into some interval on the real line.

Throughout this paper we denote by $\delta_\Omega(z)$ the following function on $\mathbf{C}^n$

$$\delta(z) = \delta_\Omega(z) = \begin{cases} -\text{dist}(z, b\Omega) \text{ if } z \epsilon \overline{\Omega} \\ \text{dist}(z, b\Omega) \text{ if } z \notin \Omega \end{cases}$$

[Here dist(z, $b\Omega$) means the euclidean distance between z and $b\Omega$.]

It is well known that the function $-\log[-\delta_\Omega(z)]$ is a plurisubharmonic exhaustion function for any bounded pseudoconvex domain Ω in $\mathbf{C}^n$. But this function is in contrast to the function ρ in condition (*1*) unbounded from above. On the other hand, not every bounded pseudoconvex domain Ω in $\mathbf{C}^n$ has property (*1*) as the following example shows: Take

$$\Omega_1 = \{(z,w) \epsilon \mathbf{C}^2 \,|\, |z| < |w| < 1\}^2$$

Assume that $\rho(z, w)$ is a plurisubharmonic exhaustion function on Ω_1 satisfying condition (*1*). Then the subharmonic function $\tilde{\rho} = \rho|\{z = 0\}$ on $\{0 < |w| < 1\}$ has a removable singularity at 0 and, because of the maximum principle, $\tilde{\rho}$ must be constant, which is a contradiction.

The same domain Ω_1 is also an example of a bounded pseudoconvex domain not satisfying condition (*2*), because any Stein domain $\hat{\Omega}$ containing $\overline{\Omega}_1$ must contain the polycylinder $\{(z, w) | \, |z| < 1, |w| < 1\}$.

Does the situation change under the additional hypothesis on Ω that $b\Omega$ is smooth? In this case the function $\delta(z)$ is also smooth and because of the plurisubharmonicity of $-\log[-\delta(z)]$ in Ω the level sets $\{z \epsilon \Omega | \delta(z) = c\}$ are smooth pseudoconvex surfaces for all negative c close to 0. Nevertheless, the function δ itself is in general not plurisubharmonic in Ω near $b\Omega$. And it can even happen that there is no smooth convex function $\varphi: (-\epsilon, 0) \to (-\infty, 0)$ such that $\varphi \circ \delta$ is plurisubharmonic near $b\Omega$. An example of this phenomenon is given by

$$\Omega_2 = \{(z,w) \epsilon \mathbf{C}^2 | Im\ w < -4(|z|^4 + |z|^2|w|^2) + 2(Re\ w)(Im\ z)\}$$

(This domain Ω_2 is unbounded, but we will be interested only in its shape near the origin.) A careful study of the Leviform of $\varphi \circ \delta$ near the origin gives:

PROPOSITION. *If φ is any strictly increasing smooth function on $(-\epsilon, 0)$ such that $\varphi \circ \delta$ is plurisubharmonic on $V \cap \Omega_2$ for some neighborhood V of the origin, then there exist positive constants K and $\hat{\epsilon} \leq \epsilon$, such that*

$$\varphi(t) > -K \log(-t)$$

for all $t \epsilon (-\hat{\epsilon}, 0)$.

As to condition (*2*), the situation is even more complicated. Namely, if one considers the pseudoconvex domain

$$\Omega_3 = \{(z,w) \epsilon \mathbf{C}^2 | Re(z - w^2) < 0\},$$

then it can be shown that the exterior level sets $\{(z, w) \epsilon \mathbf{C}^2 | \delta(z, w) = c\}$ for all $c > 0$ and small enough are strictly pseudoconcave at every point. (This is again a local phenomenon for which the unboundedness of Ω_3 is unimportant.) The result shows that the level sets of the euclidean distance do not give in general a basis of Stein neighborhoods for $\overline{\Omega}$.

Existence of bounded plurisubharmonic exhaustion functions and the Serre conjecture

In spite of the example Ω_2, smoothness of the boundary $b\Omega$ is a sufficient condition for the existence of a bounded plurisubharmonic exhaustion function, as the following theorem shows.

THEOREM 1. *Let X be a Stein manifold and $\Omega \subset X$ a relatively compact pseudoconvex domain in X with smooth boundary. Then there exists a C^∞ strictly plurisubharmonic function ρ on Ω with negative values and $\lim_{p \to b\Omega} \rho(p) = 0$.*

Moreover, ρ can always be chosen in such a way that for some large positive integer l the function ρ^l is smooth on $\bar{\Omega}$.

Remark: Locally, there always exist better exhaustion functions. For every point $p \in b\Omega$ and every $\epsilon > 0$ one can find a neighborhood U of p and a C^∞ strictly plurisubharmonic function ρ on $U \cap \Omega$ with negative values and $\lim_{p \to U \cap b\Omega} \rho(p) = 0$, such that $(-\rho)^{1+\epsilon}$ is already smooth on $\bar{\Omega} \cap U$.

In the proof of *Theorem 1* it is at first assumed that Ω lies in $\mathbf{C}^n$, and for each boundary point $p \in b\Omega$ a function ρ_p on Ω is constructed satisfying the conditions of the theorem near p. This construction depends only on the properties of $b\Omega$ near p. After this one embeds, in the general situation, X as a closed submanifold into some $\mathbf{C}^n$ and extends Ω to a bounded pseudoconvex domain $\hat{\Omega}$ in $\mathbf{C}^N$ such that $b\Omega \subset b\hat{\Omega}$ and $b\hat{\Omega}$ is smooth near $b\Omega$. A finite number of functions ρ_p for $\hat{\Omega}$ as constructed in the first step can then be used to get a function ρ on Ω with the desired properties.

As applications of *Theorem 1* we want to mention the following two statements:

THEOREM 2. *Let $\Omega \subset X$ be as in Theorem 1, and let X be embedded as a closed submanifold into some C^N. Let $\pi : U \to X$ be a holomorphic retraction from a neighborhood U of X onto X. Then there is a bounded pseudoconvex domain $\Omega' \subset\subset U$ with smooth boundary such that $\Omega' \cap X = \Omega$ and $\pi(b\Omega') = \bar{\Omega}$. The domain Ω' can be chosen to be strictly pseudoconvex outside X.*

The proof of this theorem follows the argument of H. Rossi (ref. 1).

Theorem 1 shows that the considered domains Ω are hyperconvex in the sense of J. L. Stehlé (ref. 2). As a consequence one gets at once the verification of the conjecture of J. P. Serre on holomorphic fiber bundles in the following special situation:

THEOREM 3. *Let S be the total space of a locally trivial holomorphic fiber bundle with a Stein space B as base space and typical fiber Ω. If Ω is a relatively compact pseudoconvex domain with smooth boundary in a Stein manifold X or a finite intersection of such domains in X, then S is itself a Stein space.*

Remark: In his preprint,* "Holomorphic Fiber Bundles Whose Fibers Are Bounded Stein Domains with Zero First Betti Number," Y. -T. Siu has recently shown that the Serre conjecture is true, if the typical fiber Ω is a relatively compact pseudoconvex domain with zero first Betti number in a Stein manifold X with trivial canonical line bundle. There are of course domains Ω satisfying the suppositions of *Theorem 3* with non-zero first Betti number.

Stein neighborhoods

Let Ω again be a relatively compact pseudoconvex domain with smooth boundary in a connected Stein manifold of dimension n. Let ρ be a smooth function on a neighborhood U of $b\Omega$ defining Ω, i.e.,

$$\Omega \cap U = \{p \in U | \rho(p) < 0\}$$
$$\text{and } d\rho(p) \neq 0 \text{ for all } p \in b\Omega$$

Define for $k = 0, \ldots, n - 1$ the set $M_k = \{p \in b\Omega|$ the Leviform of ρ at p with respect to some holomorphic coordinate system around p has at most k positive eigenvalues on $T_p{}^{1,0}b\Omega\}$.

Here $T_p{}^{1,0}b\Omega$ denotes as usual the holomorphic tangent space to $b\Omega$ at p. The M_k do not depend on the special choice of the function ρ; they are closed and

$$M_0 \subset M_1 \subset \ldots \subset M_{n-1} = b\Omega.$$

Then a sufficient condition for the existence of a neighborhood system of $\bar{\Omega}$ consisting of Stein neighborhoods can be formulated as follows:

THEOREM 4. *Let Ω and M_k, $k = 0, \ldots, n - 1$ be as above. Suppose that $M_k - M_{k-1}$ is a closed CR-submanifold of $b\Omega - M_{k-1}$ and that for all $p \in M_k - M_{k-1}$ the Leviform of ρ (with respect to some holomorphic coordinate system around p) is strictly positive definite on $T_p^{1,0}M_k - \{0\}$, $k = 0, \ldots, n - 2$. Then for any open set $W \supset \bar{\Omega}$, there is a strictly pseudoconvex domain $\hat{\Omega}$ with $\bar{\Omega} \subset \hat{\Omega} \subset W$.*

Remarks. (*a*) The hypothesis of *Theorem 4* on the sets M_k are in particular satisfied, if the set $M = M_{n-2}$ of points on $b\Omega$, at which $b\Omega$ is not strictly pseudoconvex, is a totally real closed submanifold of $b\Omega$.

(*b*) For all Stein domains $\hat{\Omega}$ as constructed in the proof of *Theorem 4* the pair $(\Omega, \hat{\Omega})$ is in fact Runge.

(*c*) The proof gives also that $\bar{\Omega}$ is holomorphically convex and uniformly H-convex in the sense of Čirka (refs. 3 and 4). This means that $\bar{\Omega}$ has certain nice properties with respect to approximation of differentiable functions on $\bar{\Omega}$, which are holomorphic on Ω, by holomorphic functions on $\bar{\Omega}$ (see, for instance, Čirka, refs. 3 and 4, theorem 2, p. 101).

In the proof of *Theorem 4* one reduces the statement at first to the case where Ω is contained in $\mathbf{C}^n$ by using *Theorem 2* and then constructs a positive function f on $b\Omega$, such that the points $p + f(p)\, n(p)$ describe $b\hat{\Omega}$ if p varies on $b\Omega$. Here $n(p)$ denotes the exterior normal on $b\Omega$ at p.

By using *Theorem 4* one can show:

THEOREM 5. *Let Ω be a bounded pseudoconvex set in C^2 with real-analytic boundary. Then $\bar{\Omega}$ has a neighborhood basis of strictly pseudoconvex domains.*

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